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N=4 Superconformal Algebras and Gauged WZW Models

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Abstract

As shown by Witten the $N = 1$ supersymmetric gauged WZW model based on a group G has an extended $N = 2$ supersymmetry if the gauged subgroup H is so chosen that G/H is Kähler. We extend Witten's result and prove that the $N = 1$ supersymmetric gauged WZW models over $G \times U(1)$ are actually invariant under $N = 4$ superconformal transformations if the gauged subgroup H is such that $G/(H \times SU(2))$ is a quaternionic symmetric space. A previous construction of "maximal" $N = 4$ superconformal algebras with $SU(2) \times SU(2) \times U(1)$ symmetry is reformulated and further developed so as to relate them to the $N = 4$ gauged WZW models. Based on earlier results we expect the quantization of $N = 4$ gauged WZW models to yield the unitary realizations of maximal $N = 4$ superconformal algebras provided by this construction.

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1 Introduction

Wess-Zumino-Witten models [1, 2] have been studied extensively over the last decade. They provide examples of conformally invariant field theories in two dimensions and have a very rich structure [2]. Their $N = 1$ supersymmetric extensions were introduced in [3] and they exist for any group manifold G. However, the requirement of having more than one supersymmetry imposes constraints on the possible WZW models. This is to be expected from the earlier results on supersymmetric sigma models and supergravity theories with extended supersymmetry. For example, in [4] it was shown that $N = 2$ supersymmetry requires the scalar manifold of a 2-dimensional sigma model without the Wess-Zumino term to be Kahler and $N = 4$ supersymmetry requires it to be Hyperkahler. The results of Witten [2] on the conformal invariance of WZW models for quantized values of the coefficient of WZ term provided the motivation for study of supersymmetric sigma models with such a term [5, 6, 7]. The supersymmetric sigma models relevant for heterotic strings have been studied in [8, 9]. In [10, 11] sigma models on group manifolds with extended rigid supersymmetry were studied and a classification of all such manifolds with $N = 2$ and $N = 4$ supersymmetry was given. More recently WZW models on the group manifold $SU(2) \times U(1)$ with $N = 2$ and $N = 4$ supersymmetries were studied using superspace techniques [12, 13].

In parallel to the work on sigma models a large amount of work has been done on the unitary realizations of conformal and superconformal algebras with a central charge. The Sugawara-Sommerfield contruction [14] was generalized to the coset space construction of Virasoro algebra and its $N = 1$ supersymmetric extension in [15]. Kazama and Suzuki showed that the generalization of the GKO construction to $N = 2$ superconformal algebras requires that the corresponding coset space be Kahlerian [16]. The realization of maximal $N = 4$ superconformal algebras with $SU(2) \times SU(2) \times U(1)$ symmetry [17, 18] was studied in [11, 19]. The required coset spaces for maximal $N = 4$ supersymmetry are of the form $W \times SU(2) \times U(1)$ where W is the quaternionic symmetric space associated with the group G. In [20, 21, 22] the unitary realizations of extended superconformal algebras ($N = 2$ and $N = 4$) over triple systems were given. Only for a very special class of triple systems ,called the Freudenthal triple systems (FTS), the realizations of $N = 2$ superconformal algebras (SCA) admit an extension to $N = 4$ SCA's [21, 22]. All the coset space realizations of $N = 4$ SCA's can be thus obtained from their underlying FTS's. The unitary representations of the maximal $N = 4$ superconformal algebra were studied in [23] and their characters in [24].

Even though the GKO construction and its supersymmetric generalizations provide us with a large class of unitary representations of conformal and superconformal algebras it is desirable to work with Lagrangian field theories whose quantization leads to these unitary realizations. The conformally invariant field theories corresponding to the coset models G/H are the gauged WZW models based on the group G with a gauged subgroup H. This connection was established for purely bosonic models in [25, 26] and for the $N = 1$ models in [27]. Recently, in his study of matrix models, Witten gave a simple formulation of $N = 1$ gauged WZW models in component formalism and extended it to the $N = 2$ supersymmetric theories corresponding to the Kazama-Suzuki models [28]. In this paper we generalize Witten's results to $N = 4$ supersymmetric gauged WZW models. The first part of the paper is devoted to reformulating and simplifying the construction of $N = 4$ superconformal algebras over FTS's given in [21, 22] so as to make the connection with the $N = 4$ gauged WZW models more transparent. [1]

2 A Construction of $N = 2$ Subalgebras of $N = 4$ Superconformal Algebras

As mentioned earlier a general method for the construction of extended superconformal algebras over triple systems was developed in [20, 21, 22]. For a very special class of triple systems, namely the Freudenthal triple systems (FTS), the $N = 2$ superconformal algebras thus constructed can be extended to $N = 4$ superconformal algebras with the symmetry group $SU(2) \times SU(2) \times U(1)$ [21, 22]. For FTS's with a non-degenerate symplectic form these realizations of $N = 2$ superconformal algebras are equivalent to their realization over certain Kahlerian coset spaces of Lie groups a la Kazama and Suzuki [16]. For a simple group G the associated coset space is $G/H \times U(1)$ where H is such that $G/H \times SU(2)$ is a quaternionic symmetric space as expected from the results of [11, 23]. We shall now review and reformulate this construction in a way that does not require familiarity with

[1] Throughout this paper the term $N = 4$ SCA refers to the maximal $N = 4$ superconformal algebra with $SU(2) \times SU(2) \times U(1)$ symmetry.

the underlying FTS's.

Let g, h be the Lie algebras of G and H, respectively. The Lie algebra g can be given a 5-graded structure

$$g = g^{-2} \oplus g^{-1} \oplus g^{0} \oplus g^{+1} \oplus g^{+2} \tag{2.1}$$

such that $g^0 = h \oplus K_3$ where K_3 is the generator of the $U(1)$ factor and the grade ± 2 subspaces are one dimensional. Let us denote them as

$$\begin{aligned} K_+ &= K_1 + iK_2 \in g^{+2} \\ K^+ &= K_- = K_1 - iK_2 \in g^{-2} \end{aligned} \tag{2.2}$$

The elements of the grade $+1$ and -1 subspaces will be denoted as U_a and U^a, respectively, where $a, b, .. = 1, 2, .., D$ with D being the dimension of the underlying FTS. There is a universal relation between D and the dual Coxeter number $\check{g}$ of G:

$$D = 2(\check{g} - 2) \tag{2.3}$$

Under hermitian conjugation we have

$$\begin{aligned} K_- &= K_+^\dagger \\ U^a &= U_a^\dagger \end{aligned} \tag{2.4}$$

The generators K_-, K_+ and K_3 form an $SU(2)$ subalgebra of g.

$$\begin{aligned} [K_+, K_-] &= 2K_3 \\ [K_3, K_\pm] &= \pm K_\pm \end{aligned} \tag{2.5}$$

The commutation relations of the U's are

$$\begin{aligned} [U_a, U_b] &= \Omega_{ab} K_+ \\ [U^a, U^b] &= \Omega^{ab} K_- \\ [U_a, U^b] &= S_a^b \end{aligned} \tag{2.6}$$

where Ω_{ab} is a symplectic invariant tensor of H and S_a^b are the generators of the subgroup $H \times U(1)$. The tensor Ω^{ab} is the inverse of Ω_{ab} and satisfies :

$$\begin{aligned}\Omega_{ab}\Omega^{bc} &= \delta_a^c \\ \Omega_{ab}^\dagger &= \Omega^{ba} = -\Omega^{ab}\end{aligned} \tag{2.7}$$

The trace component of S_a^b gives the $U(1)$ generator

$$K_3 = \frac{1}{2(\breve{g}-2)} S_a^a \tag{2.8}$$

Therefore we have the decomposition

$$S_a^b = H_a^b + \delta_a^b K_3 \tag{2.9}$$

where $H_a^b = S_a^b - \frac{1}{D}\delta_a^b S_c^c$ are the generators of the subgroup H. Note that H_a^b commutes with K_3, K_+ and K_-. The other non-vanishing commutators of g are

$$\begin{aligned}[][K_+, U^a] &= \Omega^{ab} U_b \\ [K_-, U_a] &= \Omega_{ab} U^b \\ [K_3, U^a] &= -\frac{1}{2} U^a \\ [K_3, U_a] &= \frac{1}{2} U_a \\ [S_a^b, U_c] &= \Sigma_{ac}^{bd} U_d \\ [S_a^b, U^c] &= -\Sigma_{ad}^{bc} U^d \\ [S_a^b, S_c^d] &= \Sigma_{ac}^{be} S_e^d - \Sigma_{ae}^{bd} S_c^e\end{aligned} \tag{2.10}$$

where Σ_{ab}^{cd} are the structure constants of the corresponding FTS which are normalized such that

$$\Sigma_{ab}^{ac} = (\breve{g}-2)\delta_b^c$$

$$
\begin{aligned}
\Sigma_{ab}^{bc} &= (\check{g}-1)\delta_a^c \\
\Sigma_{ab}^{cd} - \Sigma_{ab}^{dc} &= \Omega_{ab}\Omega^{cd}
\end{aligned}
\tag{2.11}
$$

Consider now the affine Lie algebra $\hat{g}$ defined by g. It can similarly be given a 5-graded structure with the central charge belonging to the grade zero subspace. The commutation relations of $\hat{g}$ can be written in the form of operator products as follows [21, 22]:

$$
\begin{aligned}
U_a(z)U^b(w) &= \tfrac{k\delta_a^b}{(z-w)^2} + \tfrac{S_a^b(w)}{(z-w)} + \cdots \\
U_a(z)U_b(w) &= \tfrac{\Omega_{ab}K_+(w)}{(z-w)} + \cdots \\
S_a^b(z)U_c(w) &= \tfrac{\Sigma_{ac}^{bd}U_d(w)}{(z-w)} + \cdots \\
S_a^b(z)S_c^d(w) &= \tfrac{k\Sigma_{ac}^{bd}}{(z-w)^2} + \tfrac{1}{(z-w)}(\Sigma_{ac}^{be}S_e^d - \Sigma_{ae}^{bd}S_c^e)(w) + \cdots \\
K_3(z)K_\pm(w) &= \tfrac{\pm K_\pm(w)}{(z-w)} + \cdots \\
K_+(z)K_-(w) &= \tfrac{k}{(z-w)^2} + \tfrac{2J_3(w)}{(z-w)} + \cdots \\
K_+(z)U^a(w) &= \tfrac{\Omega^{ab}U_b(w)}{(z-w)} + \cdots \\
K_-(z)U_a(w) &= \tfrac{\Omega_{ab}U^b(w)}{(z-w)} + \cdots
\end{aligned}
\tag{2.12}
$$

To construct the supersymmetry generators we also need to introduce fermi fields corresponding to the coset space $G/H \times U(1)$ [21, 22]:

$$
\begin{aligned}
&\psi_a(z)\psi^b(w) = \tfrac{\delta_a^b}{(z-w)} + \cdots \\
&\psi_+(z)\psi^+(w) = \tfrac{1}{(z-w)} + \cdots \\
&\psi_a(z)\psi_b(w) = \cdots \\
&\psi_+(z)\psi_+(w) = \cdots \\
&\psi_a(z)\psi^+(w) = \cdots
\end{aligned}
\tag{2.13}
$$

Then the supersymmetry generators are given by the following expressions [2]

$$
G(z) = \sqrt{\tfrac{2}{k+\check{g}}}\{U_a\psi^a + K_+\psi^+ - \tfrac{1}{2}\Omega_{ab}\psi^a\psi^b\psi_+\}(z) \tag{2.14}
$$

$$
\bar{G}(z) = \sqrt{\tfrac{2}{k+\check{g}}}\{U^a\psi_a + K^+\psi_+ - \tfrac{1}{2}\Omega^{ab}\psi_a\psi_b\psi^+\}(z) \tag{2.15}
$$

They satisfy

$$
\begin{aligned}
&G(z)\bar{G}(w) = \tfrac{2c/3}{(z-w)^3} + \tfrac{2J(w)}{(z-w)^2} + \tfrac{2T(w)+\partial J(w)}{(z-w)} + \cdots \\
&T(z)T(w) = \tfrac{c/2}{(z-w)^4} + \tfrac{2T(w)}{(z-w)^2} + \tfrac{\partial T(w)}{(z-w)} + \cdots \\
&J(z)G(w) = \tfrac{G(w)}{(z-w)} + \cdots \\
&J(z)\bar{G}(w) = -\tfrac{\bar{G}(w)}{(z-w)} + \cdots \\
&T(z)J(w) = \tfrac{J(w)}{(z-w)^2} + \tfrac{\partial J(w)}{(z-w)} + \cdots \\
&T(z)G(w) = \tfrac{\frac{3}{2}G(w)}{(z-w)^2} + \tfrac{\partial G(w)}{(z-w)} + \cdots \\
&T(z)\bar{G}(w) = \tfrac{\frac{3}{2}\bar{G}(w)}{(z-w)^2} + \tfrac{\partial \bar{G}(w)}{(z-w)} + \cdots
\end{aligned}
\tag{2.16}
$$

where

[2] Composite operators with a single argument are assumed to be normal ordered.

$$
\begin{aligned}
T(z) \quad = \quad & \tfrac{1}{k+\check{g}}\{\tfrac{1}{2}(U_aU^a + U^aU_a) + \tfrac{1}{2}(K_+K^+ + K^+K_+) \\
& -\tfrac{k+1}{2}(\psi_a\partial\psi^a + \psi^a\partial\psi_a) - \tfrac{1}{2}(k+\check{g}-2)(\psi_+\partial\psi^+ + \psi^+\partial\psi_+) \\
& +H_a^b\psi^a\psi_b + K_3(\psi^a\psi_a + 2\psi^+\psi_+) + \psi_+\psi^+\psi^a\psi_a + \tfrac{1}{4}\Omega_{ab}\psi^a\psi^b\Omega^{cd}\psi_c\psi_d\}(z)
\end{aligned} \tag{2.17}
$$

$$
J(z) = \frac{1}{k+\check{g}}\{2(\check{g}-1)K_3 + (k+1)\psi^a\psi_a + (k-\check{g}+2)\psi^+\psi_+\}(z) \tag{2.18}
$$

The above realization of $N = 2$ SCA's corresponds to the coset $G/H \times U(1)$ where the $U(1)$ generator K_3 determines the 5-graded structure of g. Formally, one can write

$$
[2K_3, g^m] = mg^m \tag{2.19}
$$

where g^m denotes the subspace of grade m with $m = 0, \pm1, \pm2$.

The Lie algebra g can be given a 5-graded structure with respect to K_1 as well as K_2. Therefore, one can realize the $N = 2$ SCA equivalently over the coset $G/H \times U(1)'$ or the coset $G/H \times U(1)''$, where the generators of $U(1)'$ and $U(1)''$ are K_1 and K_2, respectively. The grade ±1 and ±2 subspaces with respect to K_1 are

$$
\begin{aligned}
U'_a &= \frac{1}{\sqrt{2}}(U_a + \Omega_{ab}U^b) \\
U^{a'} &= \frac{1}{\sqrt{2}}(U^a - \Omega^{ab}U_b) \\
K'_+ &= i(K_1 + iK_2) \\
K'_- &= -i(K_1 - iK_2)
\end{aligned} \tag{2.20}
$$

They satisfy

$$
[U'_a, U'_b] \quad = \quad \Omega_{ab}K'_+
$$

$$
\begin{aligned}
[U^{a'}, U^{b'}] &= \Omega^{ab} K'_- \\
[K'_+, U^{a'}] &= \Omega^{ab} U'_b \\
[K'_-, U'_a] &= \Omega_{ab} U^{b'}
\end{aligned}
\tag{2.21}
$$

The grade ± 1 and ± 2 subspaces with respect to K_2 are

$$
\begin{aligned}
U''_a &= \frac{1}{\sqrt{2}}(U_a + i\Omega_{ab} U^b) \\
U^{a''} &= \frac{1}{\sqrt{2}}(U^a + i\Omega^{ab} U_b) \\
K''_+ &= -i(K_3 + iK_1) \\
K''_- &= i(K_3 - iK_1)
\end{aligned}
\tag{2.22}
$$

with the commutation relations similar to those of equations (2.21).

For every simple Lie group G (except for $SU(2)$) there exists a subgroup $H \times U(1)$, unique up to automorphisms, such that g has a 5-graded structure with respect to the subalgebra $h \oplus K$, where K is the generator of the $U(1)$ subgroup. Below we list the simple groups G together with their subgroups H:

G	H
$SU(n)$	$U(n-2)$
$SO(n)$	$SO(n-4) \times SU(2)$
$Sp(2n)$	$Sp(2n-2)$
G_2	$SU(2)$
F_4	$Sp(6)$
E_6	$SU(6)$
E_7	$SO(12)$
E_8	E_7

3 The Construction of $N = 4$ Superconformal Algebras

The commutation relations of the $N = 4$ SCA, denoted as $\mathcal{A}_\gamma$, with the gauge group $SU(2) \times SU(2) \times U(1)$ and four dimension 1/2 generators can be written as the following operator products relations [17][3]

$$\begin{aligned}
G_a(z)G_b(w) &= \frac{2c}{3}\delta_{ab}(z-w)^{-3} + (z-w)^{-2}2M_{ab}(w) \\
&\quad + (z-w)^{-1}[2T(w)\delta_{ab} + \partial M_{ab}(w)] + \cdots \\
M_{ab} &= \frac{4}{(k^+ + k^-)}[k^- \alpha^{+i}_{ab} V^+_i + k^+ \alpha^{-i}_{ab} V^-_i] \\
V^{\pm i}(z)G_a(w) &= \alpha^{\pm i}{}_a{}^b [G_b(w)\ (z-w)^{-1} \mp \frac{2}{k^+ + k^-} k^\pm\, \xi_b(w)\ (z-w)^{-2}] + \cdots \\
V^{\pm i}(z)\, V^{\pm j}(w) &= i\varepsilon^{ijk} V^{\pm_k}(w)\ (z-w)^{-1} - \frac{k^\pm}{2}\delta^{ij}(z-w)^{-2} + \cdots \\
\xi_a(z)\, G_b(w) &= \left[\frac{-2i}{\sqrt{k^+ + k^-}}(\alpha^{+i}_{ab} V^+_i(w)\ - \alpha^{-i}_{ab} V^-_i(w)) - \frac{\delta_{ab}}{\sqrt{2}} Z(w)\right](z-w)^{-1} + \cdots \\
V^{\pm i}(z)\, \xi_a(w) &= \alpha^{\pm i}{}_a{}^b \xi_b(w)\ (z-w)^{-1} \\
Z(z)\, G_a(w) &= -\sqrt{2}\xi_a(z-w)^{-2} + \cdots \\
\xi_a(z)\, \xi_b(w) &= \frac{1}{2}\delta_{ab}(z-w)^{-1} + \cdots \\
Z(z)\, Z(w) &= (z-w)^{-2} + \cdots \\
a, b, .. &= 1, 2, 3, 4 \\
i, j, .. &= 1, 2, 3
\end{aligned} \tag{3.1}$$

plus the usual operator products of the Virasoro generator $T(z)$ with itself and the other generators. The $\alpha^{\pm i}$ are 4×4 matrices satisfying

$$\begin{aligned}
[\alpha^{\pm i}, \alpha^{\pm j}] &= i\epsilon^{ijk}\alpha^\pm_k \\
[\alpha^{+i}, \alpha^{-j}] &= 0 \\
\{\alpha^{\pm i}, \alpha^{\pm j}\} &= \frac{\delta^{ij}}{2}
\end{aligned} \tag{3.2}$$

[3]Note that our conventions and normalizations differ from those of references [11, 17, 23, 19]. The generators $(G_a, T, V^{\pm i}, Z, \xi^a)$ appearing above are all hermitian operators.

The $V^{\pm i}(z)$ are the currents of the $SU(2)^+ \times SU(2)^-$ symmetry and $Z(z)$ is the $U(1)$ current. The ξ_a are the four dimension 1/2 generators of the algebra. The central charge of the $N = 4$ SCA is simply

$$c = \frac{6k^+k^-}{k^+ + k^-} \tag{3.3}$$

where $k^\pm$ are the levels of the two $SU(2)$ currents [11].

There are many different ways of truncating the $N = 4$ SCA to an $N = 2$ SCA. For example, any pair of operators $(G_a + iG_b)$ and $(G_a - iG_b)$ generate an $N = 2$ SCA. A "maximal" $N = 2$ supersymmetric truncation leads to a $N = 2$ SCA in semidirect sum with an $N = 2$ " matter multiplet "[23]. The matter multiplet consists of a complex current $A(z)$ and a complex fermion Q of dimensions 1 and 1/2, respectively. The generators of the $N = 2$ SCA appearing in such a truncation can be decomposed as a direct sum [23]

$$\begin{aligned} T(z) &= \hat{T}(z) + T_Q(z) \\ G(z) &= \hat{G}(z) + G_Q(z) \\ \bar{G}(z) &= \hat{\bar{G}}(z) + \bar{G}_Q(z) \\ J(z) &= \hat{J}(z) + J_Q(z) \end{aligned} \tag{3.4}$$

where $T_Q, G_Q, \bar{G}_Q$ and J_Q are bilinears of the matter multiplet

$$\begin{aligned} T_Q &= \frac{1}{2}(AA^* + \partial QQ^* + \partial Q^* Q) \\ G_Q &= A^* Q \\ \bar{G}_Q &= AQ^* \\ J_Q &= QQ^* \end{aligned} \tag{3.5}$$

The operator product of $\hat{T}, \hat{G}, \hat{J}$ with T_Q, G_Q, J_Q are regular. The "irreducible" realizations of the $N = 2$ SCA generated by $\hat{T}, \hat{G}, \hat{\bar{G}}$ and $\hat{J}$ over the coset spaces of simple Lie groups can all be obtained by the construction outlined in the previous section. To extend the $N = 2$ SCA's of the previous section to $N = 4$ SCA's one needs to introduce a matter multiplet and define two additional supersymmetry generators as well as adding the matter contribution to the first two supersymmetry generators [21, 22]. The required currents of the matter multiplet turn out to be the $U(1)$ current generated by K_3 that gives the 5-graded structure of the Lie algebra g, and an additional $U(1)$ current whose generator K_0 commutes with g together with the associated fermions which we denote as a complex fermion χ_+ and its conjugate χ^+. Then the four supersymmetry generators of the $N = 4$ SCA can be written as

$$
\begin{aligned}
\frac{1}{\sqrt{2}}(G_1 + iG_2) &= \sqrt{\tfrac{2}{k+\breve{g}}}\{U_a\psi^a + K_+\psi^+ + K_3\chi_+ \\
&\quad -\tfrac{1}{2}\Omega_{ab}\psi^a\psi^b\psi_+ - \tfrac{1}{2}\psi^a\psi_a\chi_+ - \psi^+\psi_+\chi_+\} + iZ\chi_+ \\
\frac{1}{\sqrt{2}}(G_1 - iG_2) &= \sqrt{\tfrac{2}{k+\breve{g}}}\{U^a\psi_a + K^+\psi_+ + K_3\chi_+ \\
&\quad -\tfrac{1}{2}\Omega^{ab}\psi_a\psi_b\psi^+ - \tfrac{1}{2}\psi^a\psi_a\chi^+ - \psi^+\psi_+\chi^+\} - iZ\chi^+ \\
\frac{1}{\sqrt{2}}(G_3 + iG_4) &= \sqrt{\tfrac{2}{k+\breve{g}}}\{\Omega^{ab}U_a\psi_b + K_+\chi^+ + K_3\psi_+ \\
&\quad +\tfrac{1}{2}\Omega^{ab}\psi_a\psi_b\chi_+ + \tfrac{1}{2}\psi^a\psi_a\psi_+ - \chi^+\chi_+\psi_+\} + iZ\psi_+ \\
\frac{1}{\sqrt{2}}(G_3 - iG_4) &= \sqrt{\tfrac{2}{k+\breve{g}}}\{\Omega_{ab}U^b\psi^a + K^+\chi_+ + K_3\psi^+ \\
&\quad +\tfrac{1}{2}\Omega_{ab}\psi^a\psi^b\chi^+ + \tfrac{1}{2}\psi^a\psi_a\psi^+ - \chi^+\chi_+\psi^+\} - iZ\psi^+
\end{aligned}
\tag{3.6}
$$

The Virasoro generator of $\mathcal{A}_\gamma$ is given by

$$
\begin{aligned}
T(z) &= \tfrac{1}{2}\left[Z^2 - (\chi_+\partial\chi^+ + \chi^+\partial\chi_+) - (\psi_+\partial\psi^+ + \psi^+\partial\psi_+)\right](z) \\
&\quad +\tfrac{1}{k+\breve{g}}\{\tfrac{1}{2}(U_aU^a + U^aU_a) + \tfrac{1}{2}(K_+K^+ + K^+K_+) + K_3^2 \\
&\quad -\tfrac{k+1}{2}(\psi_a\partial\psi^a + \psi^a\partial\psi_a) + H_a^b\psi^a\psi_b + \tfrac{1}{4}\Omega_{ab}\psi^a\psi^b\Omega^{cd}\psi_c\psi_d\}(z)
\end{aligned}
\tag{3.7}
$$

The generators of the two $SU(2)$ currents take the form

$$\begin{aligned}
V_3^+(z) &= K_3(z) + \tfrac{1}{2}(\psi_+\psi^+ + \chi_+\chi^+)(z) \\
V_+^+(z) &= (V_1^+ + iV_2^+)(z) = (K_+ - \psi_+\chi_+)(z) \\
V_-^+(z) &= (V_1^+ - iV_2^+)(z) = (K^+ + \psi^+\chi^+)(z) \\
V_3^-(z) &= \tfrac{1}{2}(\psi^a\psi_a + \psi^+\psi_+ + \chi_+\chi^+)(z) \\
V_+^-(z) &= (V_1^- + iV_2^-)(z) = (\psi^+\chi_+ - \tfrac{1}{2}\Omega_{ab}\psi^a\psi^b)(z) \\
V_-^-(z) &= (V_1^- - iV_2^-)(z) = (\chi^+\psi_+ - \tfrac{1}{2}\Omega^{ab}\psi_a\psi_b)(z)
\end{aligned} \tag{3.8}$$

The U(1) current of the $N = 4$ SCA is $Z(z)$ and the four dimension $\frac{1}{2}$ generators are simply the fermion fields $\psi_+(z), \psi^+(z), \chi_+(z)$ and $\chi^+(z)$. One finds that the levels of the two $SU(2)$ currents are [21, 22]

$$\begin{aligned}
k^+ &= k + 1 \\
k^- &= \check{g} - 1
\end{aligned} \tag{3.9}$$

where k is the level of $\hat{g}$. The above realization of the $N = 4$ SCA corresponds to the coset space $G \times U(1)/H$.

Interestingly. one can decouple the four dimension $\frac{1}{2}$ operators and the $U(1)$ current $Z(z)$ so as to obtain a non-linear $N = 4$ SCA [23, 29] a la Bershadsky and Knizhnik [30, 31]. The generators of this non-linear algebra take the form

$$\begin{aligned}
\tilde{T} &= T - (\frac{1}{2}ZZ + \partial\xi^a\xi_a) \\
\tilde{G}_a &= G_a + \sqrt{2}Z\xi_a - \frac{2i}{3\sqrt{(k^+ + k^-)}}\epsilon_{abcd}\xi^b\xi^c\xi^d + \frac{4i}{\sqrt{k^+ + k^-}}\xi^b(\alpha_{ba}^{+i}\tilde{V}_i^+ - \alpha_{ba}^{-i}\tilde{V}_i^-) \\
\tilde{V}^{\pm i} &= V^{\pm i} + \alpha_{ab}^{\pm i}\xi^a\xi^b
\end{aligned} \tag{3.10}$$

where the conventions for the epsilon symbol are determined by the relation

$$\alpha^{\pm iab}\alpha_i^{\pm\ cd} = \frac{1}{4}(\delta^{ac}\delta^{bd} - \delta^{ad}\delta^{bc} \pm \epsilon^{abcd}) \tag{3.11}$$

They satisfy the operator product relations

$$
\begin{aligned}
\tilde{V}^{\pm i}(z)\tilde{G}_a(w) &= \alpha^{\pm i}_{ab}\,\tilde{G}^b(w)(z-w)^{-1}+\ldots \\
\tilde{G}_a(z)\tilde{G}_b(w) &= \frac{4\tilde{k}^+\tilde{k}^-}{k^++k^-}\delta_{ab}(z-w)^{-3}+2\tilde{T}\;\delta_{ab}(z-w)^{-1} \\
&\quad +\frac{8}{k}\left(\tilde{k}^-\alpha^{+i}_{ab}\tilde{V}^+_i+\tilde{k}^+\alpha^{-i}_{ab}\tilde{V}^-_i\right)(z-w)^{-2} \\
&\quad +\frac{4}{k}\partial\left(\tilde{k}^-\alpha^{+i}_{ab}\tilde{V}^+_i+\tilde{k}^+\alpha^{-i}_{ab}\tilde{V}^-_i\right)(z-w)^{-1} \\
&\quad -\frac{8}{k}(\alpha^{+i}\tilde{V}^+_i-\alpha^{-i}\tilde{V}^-_i)_{c(a}(\alpha^{+j}\tilde{V}^+_j-\alpha^{-j}\tilde{V}^-_j)_{b)}{}^c(z-w)^{-1}.
\end{aligned}
\tag{3.12}
$$

where $(a\cdots b)$ in the last expression means symmetrization with respect to a and b.

Note that the supersymmetry generators close into bilinears of currents and the levels of the two $SU(2)$ currents $\tilde{V}^{\pm}$ are $\tilde{k}^{\pm}=k^{\pm}-1$. The central charge is

$$
\tilde{c}=c-3 \tag{3.13}
$$

The realization given above for the $N=4$ SCA leads to very simple expressions for the supersymmetry generators of the non-linear algebra

$$
\begin{aligned}
\frac{1}{\sqrt{2}}(\tilde{G}_1+i\tilde{G}_2) &= \sqrt{\frac{2}{k+\breve{g}}}U_a\psi^a \\
\frac{1}{\sqrt{2}}(\tilde{G}_1-i\tilde{G}_2) &= \sqrt{\frac{2}{k+\breve{g}}}U^a\psi_a \\
\frac{1}{\sqrt{2}}(\tilde{G}_3+i\tilde{G}_4) &= \sqrt{\frac{2}{k+\breve{g}}}\Omega^{ab}U_a\psi_b \\
\frac{1}{\sqrt{2}}(\tilde{G}_3-i\tilde{G}_4) &= \sqrt{\frac{2}{k+\breve{g}}}\Omega_{ba}U^a\psi^b
\end{aligned}
\tag{3.14}
$$

The generators of the two $SU(2)$ currents also simplify

$$
\begin{aligned}
\tilde{V}^+_+ &= K_+ \\
\tilde{V}^+_- &= K_- \\
\tilde{V}_3 &= K_3 \\
\tilde{V}^-_+ &= -\frac{1}{2}\Omega_{ab}\psi^a\psi^b
\end{aligned}
$$

$$\begin{aligned}\tilde{V}_-^- &= -\frac{1}{2}\Omega^{ab}\psi_a\psi_b \\ \tilde{V}_3^- &= \frac{1}{2}\psi^a\psi_a\end{aligned} \tag{3.15}$$

The Virasoro generator of the nonlinear algebra is then

$$\begin{aligned}T(z) = & \tfrac{1}{k+\check{g}}\{\tfrac{1}{2}(U_aU^a + U^aU_a) + \tfrac{1}{2}(K_+K^+ + K^+K_+) + K_3^2 \\ & -\tfrac{k+1}{2}(\psi_a\partial\psi^a + \psi^a\partial\psi_a) + H_a^b\psi^a\psi_b + \tfrac{1}{4}\Omega_{ab}\psi^a\psi^b\Omega^{cd}\psi_c\psi_d\}(z)\end{aligned} \tag{3.16}$$

It is clear from the above expressions for the generators that the non-linear $N = 4$ SCA is realized over the symmetric space

$$G/H \times SU(2) \tag{3.17}$$

which is the unique quaternionic symmetric space associated with G .

4 Witten's Formulation of $N = 1$ and $N = 2$ Supersymmetric Gauged WZW Models

The $N = 1$ supersymmetric WZW models were studied in [3, 10] and their gauged versions in [27]. More recently, Witten gave a simple formulation of the $N = 1$ supersymmetric WZW and gauged WZW models and generalized it to models with $N = 2$ supersymmetry [28]. In this section we review Witten's construction which we shall generalize to $N = 4$ gauged WZW models in the next section.

The WZW action at level k is given by $kI(g)$ where

$$I(g) = -\frac{1}{8\pi}\int_\Sigma d^2\sigma\sqrt{h}h^{ij}Tr(g^{-1}\partial_i g\cdot g^{-1}\partial_j g) - i\Gamma \tag{4.1}$$

with the WZ functional [1] given by [2]

$$\Gamma = \frac{1}{12\pi}\int_M d^3\sigma\epsilon^{ijk}Tr(g^{-1}\partial_i g\cdot g^{-1}\partial_j g\cdot g^{-1}\partial_k g) \tag{4.2}$$

M is any three manifold whose boundary is the Riemann surface Σ with metric h. g is a group element that maps Σ into the group G. We shall work with complex coordinates $z, \bar{z}$ and choose the metric $h_{z\bar{z}} = h^{z\bar{z}} = 1$. The supersymmetric extension $I(g, \Psi)$ of the above action is obtained by adding to it the free action of Weyl fermions Ψ_L and Ψ_R in the complexification of the adjoint representation of G [28]:[4]

$$I(g, \Psi) = I(g) + \frac{i}{4\pi} \int d^2 z Tr(\Psi_L \partial_{\bar{z}} \Psi_L + \Psi_R \partial_z \Psi_R) \tag{4.3}$$

Under the supersymmetry action the fields transform as follows

$$\begin{aligned}
\delta g &= i\epsilon_- g \Psi_L + i\epsilon_+ \Psi_R g \\
\delta \Psi_L &= \epsilon_-(g^{-1}\partial_z g - i\Psi_L^2) \\
\delta \Psi_R &= \epsilon_+(\partial_{\bar{z}} g g^{-1} + i\Psi_R^2)
\end{aligned} \tag{4.4}$$

To gauge a diagonal subgroup H of the $G_L \times G_R$ symmetry of the WZW model one introduces gauge fields $(A_z, A_{\bar{z}})$ belonging to the subgroup H. The gauge invariant action, which does not involve any kinetic energy term for the gauge fields, can be written as:

$$I(g, A) = I(g) + \frac{1}{2\pi} \int_\Sigma d^2 z Tr(A_{\bar{z}} g^{-1} \partial_z g - A_z \partial_{\bar{z}} g g^{-1} + A_{\bar{z}} g^{-1} A_z g - A_{\bar{z}} A_z) \tag{4.5}$$

The gauge transformations of the fields are defined as [28]:

$$\begin{aligned}
\delta g &= [u, g] \\
\delta A_i &= -D_i u = -\partial u - [A_i, u]
\end{aligned} \tag{4.6}$$

Let $\mathcal{G}$ and $\mathcal{H}$ be the complexifications the Lie algebras of G and H. Then $\mathcal{G}$ has an orthogonal decomposition

$$\mathcal{G} = \mathcal{H} \oplus \mathcal{T} \tag{4.7}$$

where $\mathcal{T}$ is the orthocomplement of $\mathcal{H}$.

[4] We shall restrict ourselves to models that have equal number of supersymmetries in both the left and the right moving sectors.

To supersymmetrize the gauged WZW model one introduces Weyl fermions with values in $\mathcal{T}$ minimally coupled to the gauge fields and otherwise free

$$I(g,A,\Psi)=I(g,A)+\frac{i}{4\pi}\int d^2zTr(\Psi_L D_{\bar{z}}\Psi_L+\Psi_R D_z\Psi_R) \tag{4.8}$$

It is invariant under the supersymmetry transformation laws:

$$\begin{aligned}
\delta g &= i\epsilon_- g\Psi_L+i\epsilon_+\Psi_R g \\
\delta\Psi_L &= \epsilon_-(1-\Pi)(g^{-1}D_z g-i\Psi_L^2) \\
\delta\Psi_R &= \epsilon_+(1-\Pi)(D_{\bar{z}}gg^{-1}+i\Psi_R^2) \\
\delta A &= 0
\end{aligned} \tag{4.9}$$

where Π is the orthogonal projection of $\mathcal{G}$ onto $\mathcal{H}$.

Witten showed that when the coset space G/H is Kahler the above action has $N=2$ supersymmetry. In the Kahler case $\mathcal{T}$ has a decomposition

$$\mathcal{T}=\mathcal{T}_+\oplus\mathcal{T}_- \tag{4.10}$$

where $\mathcal{T}_+$ and $\mathcal{T}_-$ are in complex conjugate representations of H. Then the action can be written in the form [28]

$$I(g,\Psi,A)=I(g,A)+\frac{i}{2\pi}\int d^2zTr(\beta_L D_{\bar{z}}\alpha_L+\beta_R D_z\alpha_R) \tag{4.11}$$

where

$$\begin{aligned}
\alpha_L &= \Pi_+\Psi_L \\
\beta_L &= \Pi_-\Psi_L \\
\alpha_R &= \Pi_+\Psi_R \\
\beta_R &= \Pi_-\Psi_R
\end{aligned} \tag{4.12}$$

with Π_+ and Π_- representing the projectors onto the subspaces $\mathcal{T}_+$ and $\mathcal{T}_-$. Denoting the chiral and anti-chiral supersymmetry generators in left and right moving sectors as $G_L, \bar{G}_L$ and $G_R, \bar{G}_R$, respectively, let us define the operators

$$\begin{aligned}
\delta_G &= \epsilon_- G_L + \epsilon_+ G_R \\
\bar{\delta}_G &= \epsilon_- \bar{G}_L + \epsilon_+ \bar{G}_R
\end{aligned} \tag{4.13}$$

where the $\epsilon_\pm$ are anticommuting Grassmann parameters.

Their actions on the fields of the theory are

$$\begin{aligned}
\delta_G g &= i\epsilon_- g\alpha_L + i\epsilon_+ \alpha_R g \\
\delta_G \alpha_+ &= -i\epsilon_- \Pi_+ \alpha_L^2 \\
\delta_G \alpha_- &= i\epsilon_+ \Pi_- \alpha_R^2 \\
\delta_G \beta_L &= \epsilon_- \Pi_- (g^{-1} D_z g - i\beta_L \alpha_L - i\alpha_L \beta_L) \\
\delta_G \beta_R &= \epsilon_+ \Pi_+ (D_{\bar{z}} g g^{-1} + i\beta_R \alpha_R + i\alpha_R \beta_R) \\
\bar{\delta}_G g &= i\epsilon_- g\beta_L + i\epsilon_+ \beta_R g \\
\bar{\delta}_G \beta_L &= -i\epsilon_- \Pi_- \beta_L^2 \\
\bar{\delta}_G \beta_R &= i\epsilon_+ \Pi_+ \beta_R^2 \\
\bar{\delta}_G \alpha_L &= \epsilon_- \Pi_+ (g^{-1} D_z g - i\beta_L \alpha_L - i\alpha_L \beta_L) \\
\bar{\delta}_G \alpha_R &= \epsilon_+ \Pi_- (D_{\bar{z}} g g^{-1} + i\alpha_R \beta_R + i\beta_R \alpha_R)
\end{aligned} \tag{4.14}$$

To prove that the action is invariant under these transformations one needs to use the facts that

$$\mathrm{Tr} ab = 0 \leftrightarrow a, b \in \mathcal{T}_+$$

$$\begin{aligned} \mathrm{Tr} ab &= 0 \leftrightarrow a, b \in \mathcal{T}_- \\ [\mathcal{T}_+, \mathcal{T}_+] &\subset \mathcal{T}_+ \\ [\mathcal{T}_-, \mathcal{T}_-] &\subset \mathcal{T}_- \end{aligned} \tag{4.15}$$

Since the theory is known to be conformally invariant it suffices to restrict oneself to checking the global supersymmetries in order to prove $N = 2$ superconformal invariance. Using the equations of motion it is straightforward to show that

$$\begin{aligned} [\delta_G, \bar{\delta}'_G] &= i\epsilon'_- \epsilon_- D_z + i\epsilon'_+ \epsilon_+ D_{\bar{z}} \\ [\delta_G, \delta'_G] &= 0 \\ [\bar{\delta}_G, \bar{\delta}'_G] &= 0 \end{aligned} \tag{4.16}$$

or equivalently

$$\begin{aligned} \{G_L, \bar{G}_L\} &= -iD_z \\ \{G_R, \bar{G}_R\} &= -iD_{\bar{z}} \\ \{G_L, G_L\} &= \{\bar{G}_L, \bar{G}_L\} = 0 \\ \{G_R, G_R\} &= \{\bar{G}_R, \bar{G}_R\} = 0 \end{aligned} \tag{4.17}$$

That the supersymmetry algebra closes only on-shell is expected since one is working in component formalism.

5 N=4 Supersymmetric Gauged WZW Models

We saw above that the existence of a second supersymmetry in a supersymmetric gauged WZW model is guaranteed when the coset space G/H is Kahlerian , i.e it admits a complex structure. Therefore one would expect that to have $N = 4$ supersymmetry one needs coset spaces with three complex structure which anti-commute with each other and form a closed algebra. This is also expected from the study of $N = 4$ supersymmetric sigma models [4, 10] and the unitary realizations of $N = 4$ superconformal algebra over quaternionic symmetric spaces [11, 19, 21, 22]. In section we have reformulated the construction of [21, 22] for $N = 2$ superconformal algebras that are extendable to $N = 4$ SCA's so as to be able to relate them to gauged supersymmetric WZW models. Let us now show that the $N = 1$ supersymmetric WZW models based on the groups $G \times U(1)$ of section 3 with a gauged subgroup H such that $G/H \times SU(2)$ is a quaternionic symmetric space have actually $N = 4$ supersymmetry.

We shall designate the generators of $G \times U(1)$ as we did in sections 2 and 3 ,i.e $K_0, K_1, K_2, K_3, U_a, U^a$ and H_a^b , where K_0 is the generator of the additional $U(1)$ factor. We normalize the generator K_0 such that

$$TrK_0^2 = TrK_1^2 = TrK_2^2 = TrK_3^2 \tag{5.1}$$

The fermions associated with the grade ± 1 subspaces of the Lie algebra of G will be denoted as ψ_a, ψ^a as before. However, the fermions associated with K_0 and K_i will be denoted as $\xi^0, \xi^i (i = 1, 2, 3)$. Then the fermions in the coset $G \times U(1)/H$ can be written as

$$\Psi = 2K_0\xi^0 + 2K_i\xi^i + U_a\psi^a + U^a\psi_a \tag{5.2}$$

for both the left and the right moving sectors. The coset $G \times U(1)/H$ can be given a Kahler decomposition so that we can write Ψ as

$$\Psi = \alpha + \beta \tag{5.3}$$

where

$$\begin{aligned} \alpha &= U_a\psi^a + K_+(\xi^1 - i\xi^2) + (K_3 + iK_0)(\xi^3 - i\xi^0) \\ \beta &= U^a\psi_a + K_-(\xi^1 + i\xi^2) + (K_3 - iK_0)(\xi^3 + i\xi^0) \end{aligned} \tag{5.4}$$

The complex structure C_3 corresponding to this decomposition acts on Ψ as follows

$$C_3\Psi = -i\alpha + i\beta \tag{5.5}$$

where the index 3 in C_3 signifies the fact in the subspace $G/H \times SU(2)$ its action corresponds to commutation with$-iK_3$. Similarly, we can give a

Kahler decomposition of the coset space $G \times U(1)/H$ which selects out K_1 or K_2. For K_1 we have:

$$\begin{aligned}
\Psi &= \alpha' + \beta' \\
\alpha' &= U'_a \psi^{a'} + (K_2 + iK_3)(\xi^2 - i\xi^3) + (K_1 + iK_0)(\xi^1 - i\xi^0) \\
\beta' &= U^{a'} \psi'_a + (K_2 - iK_3)(\xi^2 + i\xi^3) + (K_1 - iK_0)(\xi^1 + i\xi^0)
\end{aligned} \tag{5.6}$$

Under the action of the corresponding complex structure C_1 we have

$$C_1 \Psi = -i\alpha' + i\beta' \tag{5.7}$$

In the case of K_2 we have

$$\begin{aligned}
\Psi &= \alpha'' + \beta'' \\
\alpha'' &= U''_a \psi^{a''} + (K_3 + iK_1)(\xi^3 - i\xi^1) + (K_2 + iK_0)(\xi^2 - i\xi^0) \\
\beta'' &= U^{a''} \psi''_a + (K_3 - iK_1)(\xi^3 + i\xi^1) + (K_2 - iK_0)(\xi^2 + i\xi^0)
\end{aligned} \tag{5.8}$$

with the complex structure action

$$C_2 \Psi = -i\alpha'' + i\beta'' \tag{5.9}$$

The fermionic part of the action can then be written in three different ways involving the pairs $(\alpha, \beta), (\alpha', \beta')$ and (α'', β'').

$$\begin{aligned}
I(\Psi, A) &= \frac{i}{4\pi} \int d^2 z Tr(\Psi_L D_{\bar{z}} \Psi_L + \Psi_R D_z \Psi_R) \\
&= \frac{i}{2\pi} \int d^2 z Tr(\beta_L D_{\bar{z}} \alpha_L + \beta_R D_z \alpha_R) \\
&= \frac{i}{2\pi} \int d^2 z Tr(\beta'_L D_{\bar{z}} \alpha'_L + \beta'_R D_z \alpha'_R) \\
&= \frac{i}{2\pi} \int d^2 z Tr(\beta''_L D_{\bar{z}} \alpha''_L + \beta''_R D_z \alpha''_R)
\end{aligned} \tag{5.10}$$

Because of the presence of gauge covariant derivatives in the action the above relations may not appear obvious. However, the fact that the fermions ξ^0 and ξ^i are singlets of the gauge group and the fact that the complex structures' actions on the coset space $G/H \times SU(2)$ generators commute with the gauge group H action imply their validity. For each form of the action in terms of an (α, β) pair one can define a pair of supersymmetry transformations in each sector as in equations 4.14 of the previous section. Let us denote them as $(G, \bar{G}), (G'.\bar{G}')$ and $(G'', \bar{G}'')$:

$$
\begin{aligned}
(G, \bar{G}) &\leftrightarrow (\alpha, \beta) \\
(G', \bar{G}') &\leftrightarrow (\alpha', \beta') \\
(G'', \bar{G}'') &\leftrightarrow (\alpha'', \beta'')
\end{aligned}
\tag{5.11}
$$

Each pair of these operators in both sectors satisfy the $N = 2$ supersymmetry algebra given by the equations 4.17. However they are not all independent. The sum of each pair gives the manifest $N = 1$ supersymmetry generator of the model in both sectors, which we shall denote as G^0:

$$
G^0 = \frac{1}{\sqrt{2}}(G + \bar{G}) = \frac{1}{\sqrt{2}}(G' + \bar{G}') = \frac{1}{\sqrt{2}}(G'' + \bar{G}'') \tag{5.12}
$$

They satisfy

$$
\begin{aligned}
\{G^0_L, G^0_L\} &= -iD_z \\
\{G^0_R, G^0_R\} &= -iD_{\bar{z}}
\end{aligned}
\tag{5.13}
$$

We define three additional supersymmetry generators (in each sector)

$$
\begin{aligned}
G^3 &= \frac{1}{i\sqrt{2}}(G - \bar{G}) \\
G^1 &= \frac{1}{i\sqrt{2}}(G' - \bar{G}') \\
G^2 &= \frac{1}{i\sqrt{2}}(G'' - \bar{G}'')
\end{aligned}
\tag{5.14}
$$

Each one of these three supersymmetry generators anticommute with G^0 and obey

$$\begin{aligned}
\{G_L^3, G_L^3\} &= \{G_L^1, G_L^1\} = \{G_L^2, G_L^2\} = -iD_z \\
\{G_R^3, G_R^3\} &= \{G_R^1, G_R^1\} = \{G_R^2, G_R^2\} = -iD_{\bar{z}} \\
\{G_L^0, G_L^1\} &= \{G_L^0, G_L^2\} = \{G_L^0, G_L^3\} = 0 \\
\{G_R^0, G_R^1\} &= \{G_R^0, G_R^2\} = \{G_R^0, G_R^3\} = 0 \qquad (5.15)
\end{aligned}$$

as a consequence of the equations 4.17. To prove that $G^\mu(\mu = 0, 1, 2, 3)$ form an $N = 4$ superalgebra we need to further show that the $G^i(i = 1, 2, 3)$ anticommute with each other. Direct proof using their actions on the fields of the theory is quite involved. However there is a simple proof which uses the quaternionic structure of the coset $G \times U(1)/H$. We first note that the complex structures C_i obey the relation

$$C_i C_j = C_k \qquad (5.16)$$

where i, j, k are in cyclic permutations of $(1, 2, 3)$. Furthermore, the action is invariant under the replacement of Ψ by $C_i\Psi$. If we start from an action with Ψ replaced by ,say, $C_1\Psi$ then the manifest $N = 1$ supersymmetry will be generated by G^1 and the second supersymmetry generated by the Kahler decomposition with respect to the complex structure C_3 will be G^2 since $C_3C_1 = C_2$. Hence by the results of the previous section we have

$$\{G^1, G^2\} = 0 \qquad (5.17)$$

and by cyclic permutation we have

$$\{G^2, G^3\} = \{G^3, G^1\} = 0 \qquad (5.18)$$

To summarize we have shown that the four supersymmetry generators G^μ satisfy the $N = 4$ supersymmetry algebra:

$$\{G_L^\mu, G_L^\nu\} = -i\delta^{\mu\nu} D_z$$

$$
\begin{aligned}
\{G_R^\mu, G_R^\nu\} &= -i\delta^{\mu\nu} D_{\bar z} \\
\mu, \nu, ... &= 0, 1, 2, 3
\end{aligned}
\tag{5.19}
$$

Since the gauged WZW models considered above are known to be conformally invariant we have thus proven that they are invariant under $N = 4$ superconformal transformations.

If we restrict ourselves to the case $G = SU(2)$ then we have the supersymmetric WZW model on the group manifold $SU(2) \times U(1)$ with no gauge fields. The $N = 4$ supersymmetry of this model was studied in [12] using superspace techniques. In the general case, we can integrate out the four fermions ξ^a that do not couple to the gauge fields of H as well as the Abelian $U(1)$ field generated by K_0 . Then we expect the resulting theory to be invariant under the non-linear $N = 4$ SCA $\tilde{\mathcal{A}}_\gamma$.

The quantization of gauged WZW models is known to yield the coset space realizations of conformal algebras [25, 26]. Therefore the quantization of the $N = 4$ gauged WZW models we have given above is expected to give the unitary realizations of maximal $N = 4$ SCA's with $SU(2) \times SU(2) \times U(1)$ symmetry that were presented in section 3. Their quantization and the study of more general gauged WZW models with extended non-linear superconformal symmetry will be left to future investigations.

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